The potential for developing algebraic thinking from purposeful guessing and checking

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Abstract: Two contrasting problem solving episodes are presented in which a pre-algebra student uses systematic guessing and checking to solve algebra word problems with the underlying structure: "For a given *y, m,* and *b,* find *x* such that $y=mx+b$." The first episode illustrates how the student initially engages in systematic guessing and checking organized in a table (a "Guess and Check chart") to converge to the solution to a given word problem. The second episode describes the student's activity in a later problem solving session where he discovers a deterministic and essentially algebraic algorithm (linear interpolation) for solving word problems of this same form. The evolution of the student's thinking from purposeful guessing and checking to an algebraic algorithm illustrates that this activity structure can potentially offer a context in which the meaningful development of the fundamental concepts of variable, function, and rate of change can be engaged.

Introduction

Many research studies have documented that students, even those who have extensive training in formal algebraic problem solving approaches, employ informal strategies such as guessing and checking when faced with making sense of algebraic contexts (Bednarz & Janvier, 1996; Hall, Kibler, Wenger, & Truxaw, 1989; Johanning, 2004, 2007; Kieran, Boileau, & Garançon, 1996; Nathan & Koedinger, 2000; Rojano, 1996). However, the value of the activity of guessing and checking is contested in different contexts within the mathematics education literature. Some previous research has ascribed a potentially negative role to guessing and checking, for example, citing it as an obstruction to learning to formulate and solve equations using algebraic methods (Stacey & MacGregor, 2000). Another problematic aspect researchers have associated with guessing and checking is that it can be a "local tactic" that undermines the purpose of the activity of generating expressions that capture the general form of geometric or numeric patterns (Healy & Hoyles, 1999; Lannin, 2005; Mason, 1996; Radford, 2006). The arguments against guessing and checking in these contexts concern students' attempts to find a rule that fits a particular instance of the pattern instead of understanding the general relation between the members of a given pattern. On the other hand, other research describes the potential for developing an understanding of the rules that could correspond to given data from linear functions expressed in a completed tabular form through engaging young students in the "Guess My Rule" game (Carraher & Earnest, 2003; Schliemann, Carraher, & Brizuela, 2001). Other research (Bills, Wilson, & Ainley, 2006; Haspekian, 2003; Johanning, 2004, 2007; Rojano, 1996; Sutherland & Rojano, 1993) has reported that students' guessing and checking approaches (in many instances in the context of using computer spreadsheets) for solving word problems can be foundational to the construction of algebraic knowledge. Such work is generally concerned with the development of meaning for algebraic symbolism through work with specific numerical instantiations of relationships described in problem contexts before algebraic notation is introduced. This instructional approach directly responds to the literature that highlights the complexity of the concept of variable and the difficulty that students have in developing meaning for symbolic algebraic notation (Heck, 2001; MacGregor & Stacey, 1997; Philipp, 1992; Schoenfeld & Arcavi, 1988). However, beyond the potential for developing meaning for algebraic symbolism, it appears that the activity of solving algebra word problems using "guessing and checking" has largely been unexplored as a potential resource to build upon in constructing algebraic knowledge around how variables co-vary in functional relationships and the rate of change of functions.

In this paper, I discuss two contrasting episodes of a pre-algebra student, Liam, using purposeful and systematic guessing and checking recorded in a chart (a "Guess and Check chart") to solve algebra word problems with the underlying structure: "For a given *y, m,* and *b,* find *x* such that $y=mx+b$." The first episode provides a baseline view of the student engaged in purposeful guessing and checking. In this initial episode, the student deliberately uses the results from previous guesses to inform subsequent guesses. Over the course of the sessions from which the focal episodes are extracted, Liam's purposeful guessing and checking strategy unfolded into the discovery of a deterministic and essentially algebraic algorithm for finding the solution to any of the word problems from the sessions. The second episode examines how Liam uses his algorithm to solve a particular word problem. In the second episode, inputs and corresponding outputs from trial values are treated as data about the underlying linear function in the problem context. Using the data generated from two trial input/output pairs, Liam is able to predict the correct next "guess." Liam's algorithm hinges upon determining the rate of change between the two trial input/output pairs. Knowing the rate of change allows him to establish how much the output would increase or decrease when the input was changed by one. Hence, starting from either of the trial input/output pairs, Liam can figure out how to alter the input value so that it achieves the target output.

Data Collection & Methodology

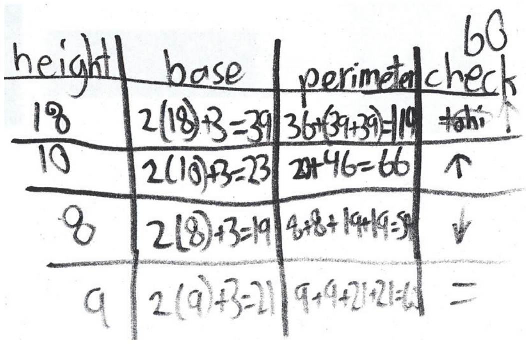
Liam (a pseudonym) was a pre-algebra student (age 13), who had just completed a year-long traditional pre-algebra course. Liam enjoyed mathematics and approached solving the problems in the sessions in remarkably intuitive and reflective ways. He was unusually articulate in discussions of his work. He frequently took the opportunity to check his work, reflect on his solution process, and was very interested in efficiency. Liam participated in a total of six "tutorial-style" sessions, each approximately one hour in length. In these sessions, he solved multiple word problems of similar underlying structure. The first session was a pre-assessment and the last session was a post-assessment. The contrasting episodes examined in this paper come from session two and session five. "Episode One" examines how Liam uses a Guess and Check chart to organize a trial and error solution strategy for solving a particular word problem. "Episode Two" is taken from a later problem solving session, during which Liam articulates an algorithm for determining the solution to any of the word problems he has been given. For each of the six sessions, both video and audio records were collected and student work artifacts were saved for analysis. Audio from all six sessions was transcribed and the resulting transcript was annotated with the relevant gestures and actions upon chart forms using the corresponding video records and student work artifacts. Nine episodes over the course of the six sessions were identified as relevant to Liam's refinement of the "guessing and checking" approach to solving problems.

The instructional content of the sessions was designed to parallel instruction described in the CPM (College Preparatory Mathematics) curriculum (Kysh, Sallee, & Hoey, 2003). CPM is a "reform-oriented" curriculum that was designed to align with the *Principles and Standards for School Mathematics* published by the National Council of Teachers of Mathematics (NCTM, 2000) in the United States. Solving word problems by organizing guessing and checking in charts is initially encouraged in the CPM curriculum. Guessing and checking in chart forms is positioned as a way to develop meaning for algebraic symbolism before students are introduced to equation-based approaches for solving word problems. The overarching goal of the tutorial sessions was to explore on a small scale with individual students over a medium length of time, how the transition from guess and check chart-based problem solving strategies to equation-modeling strategies could be supported when following the materials and guidelines in the CPM curriculum. I was influenced in the design and implementation of the instructional sessions by my experience the previous academic year as a participant-observer in a classroom where Guess and Check charts were used. The data that is presented in this paper (Liam's discovery of a deterministic and essentially algebraic algorithm for solving the word problems in the sessions) was not a planned part of the instructional activities.

EPISODE ONE: Liam engages in purposeful "guessing and checking"

At the beginning of the sessions, Liam was using informal methods, including guessing and checking, to solve word problems. From the very beginning, Liam was making purposeful decisions about how to choose his next guesses as he solved problems. During the second instructional session, I asked Liam to start recording his sequences of trial solutions in a chart form. I introduced some basic conventions that I wanted him to follow when constructing such charts. The conventions included naming a column for each of the quantities that the problem statement gives relational information about (in this case, the height and the base of the rectangle), including a column for the given constraint on these quantities (in this case, the perimeter of the rectangle), and including a column to denote whether a given trial value was too low or too high. We decided that an "up arrow" would mean that the trial value was too high and a "down arrow" would mean that the trial value was too low. The following data excerpt comes from session two and is the first time Liam solved a word problem using a guess and check chart. The problem he was working on was:

The base of a rectangle is 3 more than twice the height. If the perimeter of the rectangle is 60 inches, find the height and the base of the rectangle.



Height	base	Perimeter	60 check
18	2(18)+3=39	36+(39+39)=114	↑
10	2(10)+3=23	20+46=66	↑
8	2(8)+3=19	8+8+19+19=54	↓
9	2(9)+3=21	9+9+21+21=60	=

Figure 1**.** Liam's (original) Guess and Check chart for episode one and a typed version of his work

In terms of the underlying linear function, this problem would be: "Find x so that $6x+6=60$." In working on this problem, Liam chooses an initial guess of 18 inches. He considers changing it to 20, noting that he knows it's a bit too high, but then decides to go with his initial guess of 18. In the middle of the calculations for the perimeter of a rectangle with a height of 18 inches he notes "Oh. So it's actually a lot lower. I just realized that." As he continues to write out the calculation for 18, he conjectures that the height is 6 inches, noting "So basically, you try to double this [the height] then double that [the base] then add them [the doubled height and doubled base] to try to get 60." "This is way too much" he says as he is finishing the calculation for 18. The interviewer asks him about how much too high it was. He says, "Uh, it's almost twice as much. So I'll try it with 10." After working through the calculation for 10, he concludes that it is a little too high and says that he wants to try it with 8. He writes out the calculations for a height of 8 in the chart, concluding [since 8 is too low] "It's probably nine." When the interviewer asks him how he knew this, Liam says "Well, it was actually definitely 9 if this [the result for a guess of 8] was too low and this [the result for a guess of 10] was too high. Unless it was a decimal number."

The above episode is illustrative of some important ways in which Liam's initial activity with Guess and Check charts differs from the activity of a student engaged in "un-purposeful" guessing and checking. If we consider just the first guess of 18 that Liam specified for the height of the rectangle, there are several possible rationales for choosing a next guess. For example,

1. He could notice that he gets 114 inches for the perimeter and then decide to simply choose any other number that is not 18 as a next guess. (This would be an example of "un-purposeful" guessing and checking).
2. He could notice that 114 is greater than 60 and from this decide to choose a next guess that is less than 18.
3. He could notice that 114 is *a lot* greater than 60 and from this decide to choose a next guess that is *a lot* smaller than 18.
4. He could notice that 114 is about twice as much as 60 and decide that he should reduce his guess of 18 by roughly a factor of 2.
5. He could notice that 114 is a little less than twice as much as 60 and decide that he should reduce his guess of 18 by a little less than a factor of 2.

Note that criterion 5 includes a scalar judgment (a little more than) in addition to noticing a proportional relationship (twice as much) and is also consistent with choosing 10 as a next guess rather than 8. In the above episode, Liam may have used a criterion similar to either the fourth or fifth criterion listed above when choosing his next guess of 10. Now observe the last three rows in the chart that Liam constructed above. Liam explained that after he had found that a guess of 10 yielded a value for the perimeter that was too high and that a guess of 8 gave a perimeter that was too low, he knew that the correct value for the height would be "probably nine." This "sandwiching" idea is yet another example of how a student might use information from previous guesses to inform the choice of a next guess. For each trial value, he considers both how far off and in which direction from the target value he was each time. Additionally, considerations about how far off he was from the target value may be given in both scalar and proportional terms. This discussion is meant to demonstrate that a student might be making a number of mathematically consequential observations in the course of choosing a next guess.

EPISODE TWO: Liam discovers a deterministic, algebraic algorithm

The second focal episode is taken from the fifth instructional session. Between session two and session five, Liam had more experience solving algebra word problem using guessing and checking and recording his sequence of guesses in the charts. In the intervening sessions, we had also been considering how to encapsulate the meaning of the relationships captured in the chart using symbolic unknowns and formulating equations. This usually involved first solving the problems using guessing and checking recorded in the charts and then thinking about how to use this to formulate an appropriate equation to solve. The emergence of the algorithm that we explore in this section was not a planned part of the instructional intervention. In this episode, Liam is working on the following problem:

The sum of three consecutive integers is 414. Find the three integers.

The problem, in terms of the underlying linear function, would be: "Find x so that $3x+3=414$." In this episode, Liam's artifacts have been modified so that the reader can see what his work looked like as he was solving the problem and describing his thinking.

1. Liam I'm gonna guess one hundred-thirty-five.
2. Interviewer Okay.

(There is a pause in talking as Liam is engaged in making the chart template with columns labeled "1", "2", "3", "Sum", "Ch" and labeling his "target" of 414 above the "Sum" label.)

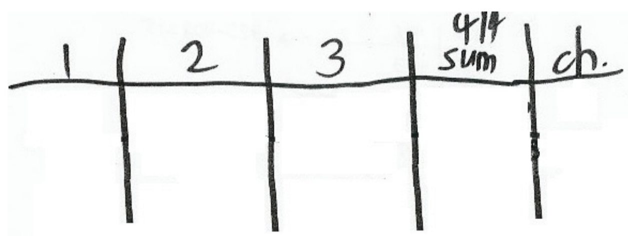


Figure 2. Liam's modified Guess and Check chart, 1

3. Interviewer Okay.
4. Liam 408. So, that's a bit too low.
5. Interviewer Okay.
6. Liam I guess 140. And the number that I'm aiming for is… Wait, you could do that when you have two… [rows], okay 140.

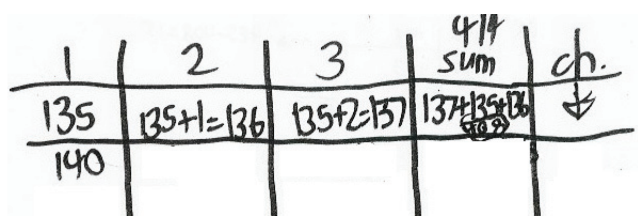


Figure 3. Liam's modified Guess and Check chart, 2

7. Interviewer So, what were you thinking of doing right there?

8. Liam Well, compare the – like how much off this number was [indicates 408, the result of his first calculation], and compare it to a different guess [indicates 140 and then the whole second row (currently blank except for 140 in the "1" column) possibly referring to the process of working through the calculation with 140] and look at that [points to currently blank entry in the sum column in the second row], see how much off it was, then you could get this [pointing to the sum of 414 in the problem statement].

9. Interviewer Okay, and what did you say=

10. Liam Like the difference between this result [pointing to 408 in sum column] and this result [indicating the second row of sum column, which is currently blank] and see how far apart these [pointing to the "inputs" 135 and 140] were, so you could tell how many of these [pointing to the entries in the "1" column] equaled how many of these [pointing to the entries in the "Sum" column].

In the above exchanges, Liam restates his approach, now in terms of the particular numbers that he has chosen. Liam is focusing attention on relationships between input/output pairs from the column labeled "1" (the input) and the column labeled "Sum" (the output). His gestures also indicate that he is concerned with the number that occupies this position rather than focused on the particular value that happens to be currently occupying that position. This is an important point—he is describing a general algorithm that involves comparing input/output pairs, not specific numerical calculations.

In lines 11-18 (not displayed), the interviewer asks again for clarification about what Liam wants to do. Liam says "I remember how to do it with two of them." The interviewer (who is still trying to understand Liam's proposed approach) interprets this as a reference to a previous problem that Liam has solved with a sum of two consecutive odd integers, not to a further elaboration of the idea of using the inputs and outputs recorded in two rows to predict the value of the first integer that will achieve the sum of 414. [Upon later examination of the transcript, it is clear that Liam is describing how to use the rate of change to predict the value of the first integer that will achieve the sum of 414]. The interviewer suggests that for the moment they just continue with the calculation for 140. In turn 19, Liam has completed the calculation for 140 as the first number.

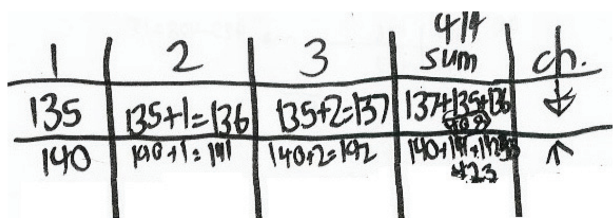


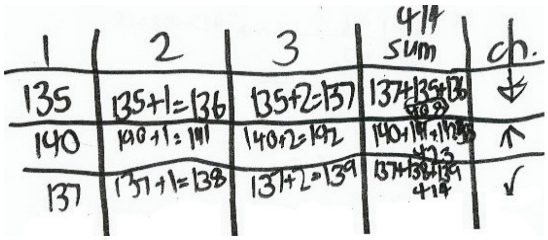
Figure 4. Liam's modified Guess and Check chart, 3

19. Liam Yeah. *(having verified the calculations for a guess of 140 with the calculator)*. That's a little bit too high.

20. Interviewer Okay.

21. Liam I said that was 408 [results from input of 135] and this 423 [results from input of 140].
The difference of this is [gesturing between 408 and 423 in the "Sum" column]—it's 15, [now moves hand over to column labeled "1"].
So for every one of these [points to entries in the "1" column], it's 3 there [points to "Sum" column].
Okay, so I'm trying to get 414.
So… *(pauses to think)*.
Ah. The difference between these two [indicates 423, the result of the calculation for 140, and 414, the target value] is 9,
so this should be less [points to 140 in the "1" column].

In turn 21, Liam is now explaining how he is trying to figure out how to predict the answer while he is the process of working with specific numbers. He is mapping from his "general" idea of how this process should work, to how it works in the case with these particular values.



1	2	3	414 Sum	Ch
135	135+1=136	135+2=137	137+135+135=408	↓
140	140+1=141	140+2=142	140+141+142=423	↑
137	137+1=138	137+2=139	137+138+139=414	=

Figure 5. Liam's final (original) Guess and Check chart with a typed version of his work

There is a pause in talking as Liam completes the next row of the chart using his predicted guess of 137 (since he had decided in turn 21 that he needed to adjust his guess by lowering it from 140 to 137). In turns 22-25, there is a brief exchange about whether his calculations for 137 achieved 414.

26. Interviewer — Good. So can you recap your reasoning? How did you decide that 137 was the best guess?

27. Liam — Well, it isn't really a guess once you get to the third one! [gesturing between the entries in the "sum" column]
I took 408 and 423. [pointing to 408 and 423 in the sum column, as he is explaining]
I have the difference between those [between 423 and 408], which is 15.
The difference between these two [pointing to 135 and 140 in the "1" column] is 5,
and 15 divided by 5 is 3.
So, that means that for every one that this [pointing to 140 in the "1" column] changes, [moves hand over to the "sum" column] the answer changes 3.
So, then I took 423 and I subtracted that [moves hand up to problem statement to indicate the target value of the sum: 414]; the difference was 9.
3 times 3 is nine, [moves hand over to 140 in the "1" column] so I knew it would have to be 3 less than this [pointing again to 140 in the "1" column], and yeah.

In contrast to the purposeful activity described in episode one, Liam has developed a more sophisticated, interpolation approach to solving the problem in the second episode. He first computes the outputs that correspond to two test values. He then uses this data to determine the rate of change between the input-output pairs. At this point, he chooses to determine the absolute difference between a particular output (here, he chose the second output) and the target output. By dividing the absolute difference by the rate of change, he can determine how many units he must adjust the input by in order to achieve the target output.

Liam's comment "It isn't really a guess once you get to the third one!" at the beginning of turn 27 is very telling. It gives evidence that already at this point, Liam understands that his approach gives a way to determine solutions to a certain class of word problems, not just find the answer to this particular word problem. It is worth noting that after the episode described here, Liam spontaneously deploys his new, general algorithm as he determines that it is appropriate when he is solving future problems using a Guess and Check chart. The awareness of the generality of applicability of the approach is an extremely important distinction between Liam's activity in episode one and his activity in episode two. Liam's activity in episode one is dependent on the continuing refinement of previously specified guesses as he converges to the solution to the word problem. In contrast, in episode two Liam has found a general method for finding the solution to any of the word problems in the sessions. Though Liam is still performing computations on particular test values, the choice of the particular values is not important. Liam has discovered that what is important is that he can figure out the rate of change between the two test values he chooses.

Discussion and Conclusions

The aim of this paper has been to highlight opportunities for the development of algebraic thinking around the concepts of variable, function, and rate of change from purposeful guessing and checking activities. The focal episodes described in this paper show that Liam's activity evolved from "guessing and checking" to "linear interpolation." At the beginning of the sessions, the guessing and checking activity was focused around determining whether or not a particular guess or trial value satisfied the constraints given in the problem and then using information about the results of that guess to inform the choice of the next trial value. This approach led to a sequence of guesses that would eventually converge to the solution to the problem. At the end of the sessions, the guessing and checking activity had evolved into using particular test values to collect data about the rate of change of the underlying linear function given in the problem contexts. Liam then used this data (in an optimal way) to accurately determine solutions to algebra word problems. Furthermore, Liam was aware of the power of his method and confident in its applicability and use in other similar contexts.

A fundamental aspect that underlies the work presented here is the role of the chart structure in mediating Liam's guessing and checking activity and the construction of his algorithm. Organizing a record of previous trials using the structure and organization of Guess and Check charts evidently supported both Liam's initial activity of refining a sequence of guesses and also Liam's later activity involving using information about how the inputs and outputs co-vary to determine solutions to the word problems. The meaning that can be constructed for variable, function, and rate of change through the examination of the patterns between particular numerical instantiations could easily be suppressed or overlooked in an activity involving guessing and checking that did not include organizing this information in a chart form. At a basic level, the use of the "check" column to record whether a guess was too high or too low distinguishes the activity with the charts from un-purposeful guessing and checking activity where one might only note whether a particular guess was right or wrong. Further research is needed to understand the nature of the support that Guess and Check charts can provide in word problem solving activities. In addition to the affordances of the structural organization of the chart, it seems likely that the activity itself of solving a problem by iterative guessing and checking supported Liam's coordination between the inputs and outputs that he was recording in the table as he solved the problem.

The research in this paper indicates that the context of a Guess and Check chart for organizing purposeful guessing and checking word problem solving strategies has potential for supporting the development of algebraic thinking around the concepts of variable, function, and rate of change. Upon reviewing Liam's activity presented in this paper, one might well wonder what the prospects for other students would be if they were to engage in similar activities. It would be premature and unwarranted to expect that other students would necessarily have the same experience as Liam. Future work examining the experiences of other students as they work with Guess and Check charts to solve algebra word problems is needed. However, especially because Liam's discovery of this algorithm is an impressive achievement, it would be of significant interest to study in more detail how he came to construct this algorithm in order to inform future instructional design.

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Acknowledgments

I would like to thank Alan Schoenfeld, Andy diSessa, Randi Engle and the members of the functions research group at UC Berkeley for their support and feedback. The research in this paper is based in part on work supported by the National Science Foundation Grant No. ESI-0119732 to the Diversity in Mathematics Education Center for Learning and Teaching.